Attribute-Efficient Learning of Monomials over Highly-Correlated Variables

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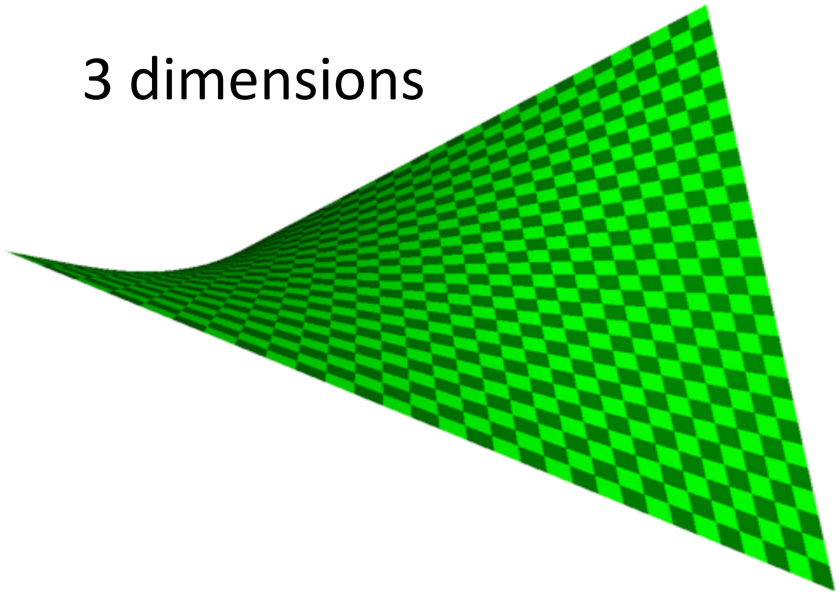
Learning Sparse Monomials

A Simple
Nonlinear
Function Class



In p dimensions
and k sparse

Ex: $f(x_1, \ldots, x_p) := \underbrace{x_3 \cdot x_{17} \cdot x_{44} \cdot x_{79}}_{k\,=\,4}$

The Learning Problem

Given: $\left\{\left(\boldsymbol{x}^{(i)}, f\left(\boldsymbol{x}^{(i)}\right)\right)\right\}_{i=1}^{m}$, drawn i.i.d.

Assumption 1: f is a k-sparse monomial function

Assumption 2: $\boldsymbol{x}^{(i)} \sim \mathcal{N}(0, \Sigma)$

Goal: Recover f exactly

Attribute-Efficient Learning

- Sample efficiency: $m = \mathrm{poly}(\log(p), k)$

- Runtime efficiency: $\mathrm{poly}(p, k, m)$ ops

- Goal: achieve both!

Motivation

$x_i \in \{\pm 1\}$

- Monomials $\equiv$ Parity functions
- No attribute-efficient algs!
  [Helmbold+ '92, Blum'98, Klivans&Servedio'06, Kalai+'09, Kocaoglu+'14…]

$x_i \in \mathbb{R}$

- Sparse **linear** regression
  [Candes+'04, Donoho+'04, Bickel+'09…]
- Sparse sums of monomials
  [Andoni+'14]

For **uncorrelated** features:

$$\mathbb{E}[xx^T] = \begin{bmatrix} \sigma_1^2 & & & & & \\ & \sigma_2^2 & & & 0 & \\ & & \sigma_3^2 & & & \\ & & & \sigma_4^2 & & \\ & 0 & & & \sigma_5^2 & \\ & & & & & \sigma_6^2 \end{bmatrix}$$

Motivation

$x_i \in \{\pm 1\}$

- Monomials
- No attribut

[Helmbold+ '92, B
Kalai+'09, Kocaog

$x_i \in \mathbb{R}$

regression

+'04, Bickel+'09...]

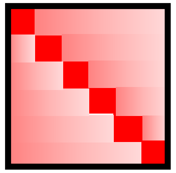
omials

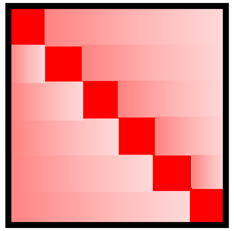
ated features:

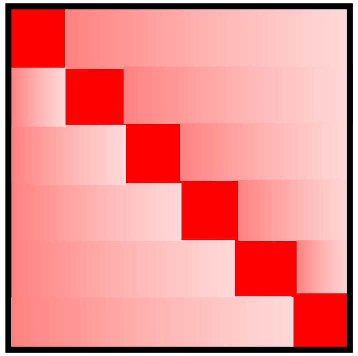
$$\begin{bmatrix} \sigma_1^2 & & & & & \\ & \sigma_2^2 & & & 0 & \\ & & \sigma_3^2 & & & \\ & & & \sigma_4^2 & & \\ & 0 & & & \sigma_5^2 & \\ & & & & & \sigma_6^2 \end{bmatrix}$$

Question: What if

$$\mathbb{E}[xx^T] = \begin{bmatrix} 1 & & & & & \\ & 1 & & \leq \rho & & \\ & & 1 & & & \\ & & & 1 & & \\ & \leq \rho & & & 1 & \\ & & & & & 1 \end{bmatrix} \ ?$$

Potential Degeneracy of  $= \mathbb{E}[xx^T]$

Ex: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_{\color{red}p} \end{bmatrix} \sim \begin{bmatrix} \mathcal{N}(0,1) \\ \mathcal{N}(0,1) \\ (x_1 + x_2)/\sqrt{2} \\ \mathcal{N}(0,1) \\ \vdots \\ \mathcal{N}(0,1) \end{bmatrix} \longrightarrow$  $= \underbrace{\begin{bmatrix} 1 & 0 & \sqrt{.5} & & & \\ 0 & 1 & \sqrt{.5} & & \mathbf{0} & \\ \sqrt{.5} & \sqrt{.5} & 1 & & & \\ & & & \ddots & & \\ & \mathbf{0} & & & 1 & 0 \\ & & & & 0 & 1 \end{bmatrix}}_{\text{Singular matrix}}$

 can be **low-rank!**

Rest of the Talk

1. Algorithm

2. Intuition

3. Analysis

4. Conclusion

1. Algorithm

The Algorithm

Ex: $f(x_1, \dots, x_p) \coloneqq x_3 \cdot x_{17} \cdot x_{44} \cdot x_{79}$

Step 1

$$\left\{\left(\boldsymbol{x}^{(i)}, f\left(\boldsymbol{x}^{(i)}\right)\right)\right\}_{i=1}^{m} \xrightarrow{\log|\cdot|} \left\{\left(\log\left|\boldsymbol{x}^{(i)}\right|, \log\left|f\left(\boldsymbol{x}^{(i)}\right)\right|\right)\right\}_{i=1}^{m}$$

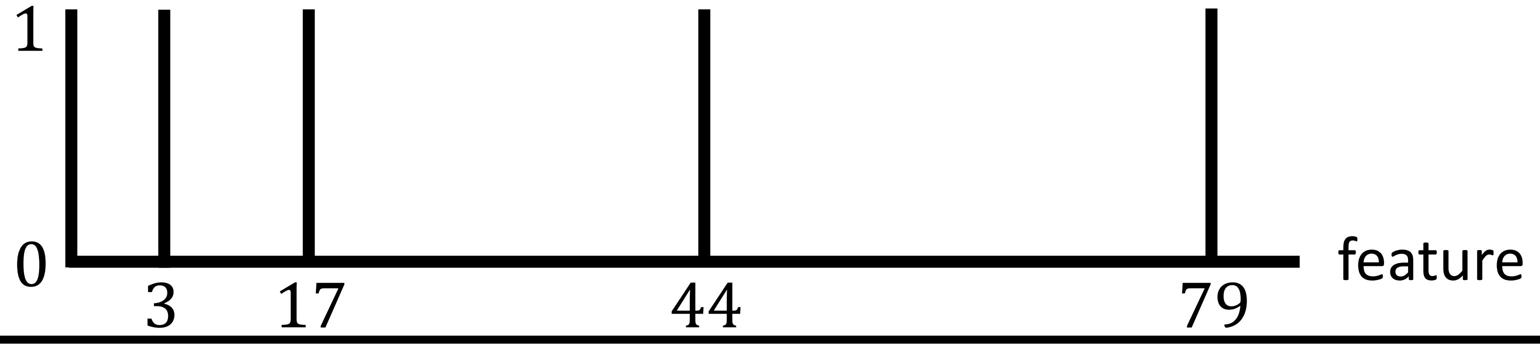
Gaussian Data

Log-transformed Data

Step 2

Sparse Regression:

(Ex: Basis Pursuit)



2. Intuition

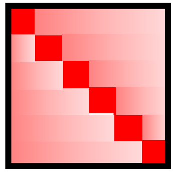
Why is our Algorithm Attribute-Efficient?

- Runtime: basis pursuit is efficient

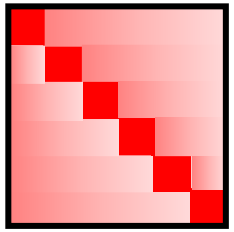
- Sample complexity?
  - Sparse **linear** regression? E.g.,

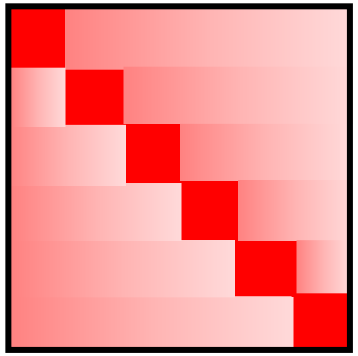
$$\log\left|f(x_1, \dots, x_p)\right| := \log|x_3| + \log|x_{17}| + \log|x_{44}| + \log|x_{79}|$$

  - But: sparse recovery properties may not hold…

 $= \mathbb{E}[xx^T]$

Degenerate High Correlation

Recall the example:  $= \begin{bmatrix} 1 & 0 & \sqrt{.5} & & & \\ 0 & 1 & \sqrt{.5} & & \mathbf{0} & \\ \sqrt{.5} & \sqrt{.5} & 1 & & & \\ & & & \ddots & & \\ & \mathbf{0} & & & 1 & 0 \\ & & & & 0 & 1 \end{bmatrix}$

 $\begin{bmatrix} -1/2 \\ -1/2 \\ 1/\sqrt{2} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{0}$

3-sparse

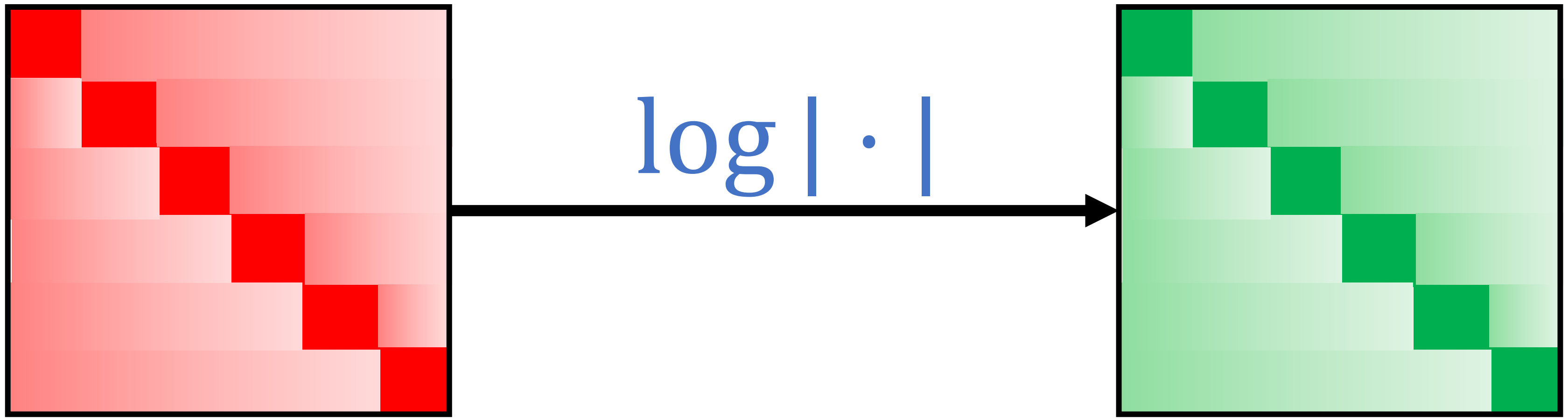
0-eigenvectors can be k-sparse

Sparse recovery conditions false!

Summary of Challenges

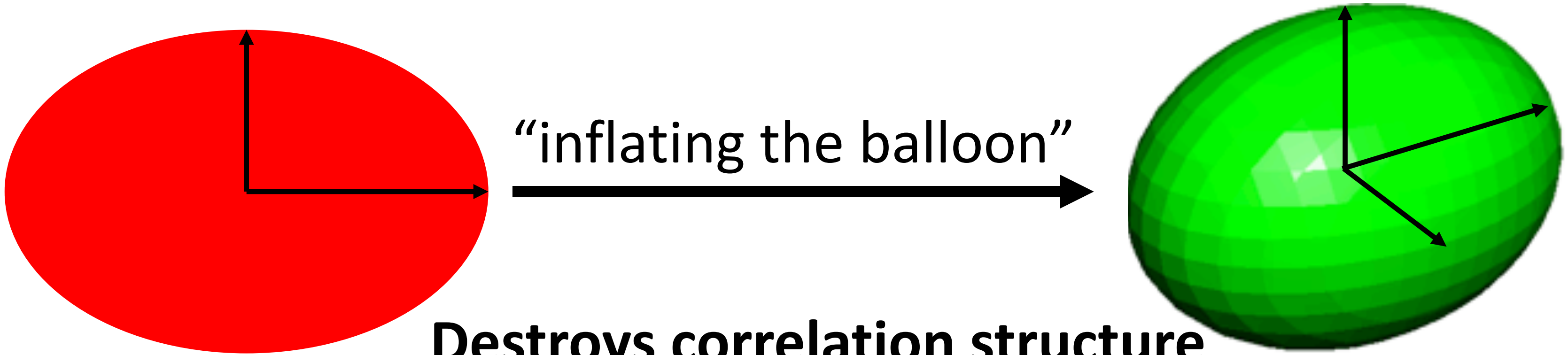
- Highly correlated features
- Nonlinearity of $\log|\cdot|$
- Need a recovery condition…

Log-Transform affects Data Covariance



$\mathbb{E}[xx^T] \succcurlyeq 0$ $\qquad$ $\mathbb{E}[\log|x| \log|x|^T] \succ 0$

Spectral View:



Destroys correlation structure

3. Analysis

Restricted Eigenvalue Condition [Bickel, Ritov, & Tsybakov ‘09]

Restricted Eigenvalue $RE(k)$

$$\min_{v \in C} \frac{v^T X X^T v}{||v||_2^2} > \epsilon$$

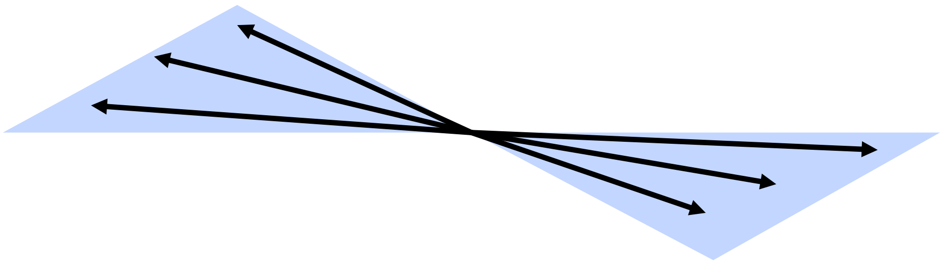
“restricted strong convexity”

Note: $RE(k) \geq \lambda_{min}(XX^T)$

Sufficient to prove exact recovery for basis pursuit!

Ex: $S = \{3, 17, 44, 79\}$
$k = 4$

Cone restriction

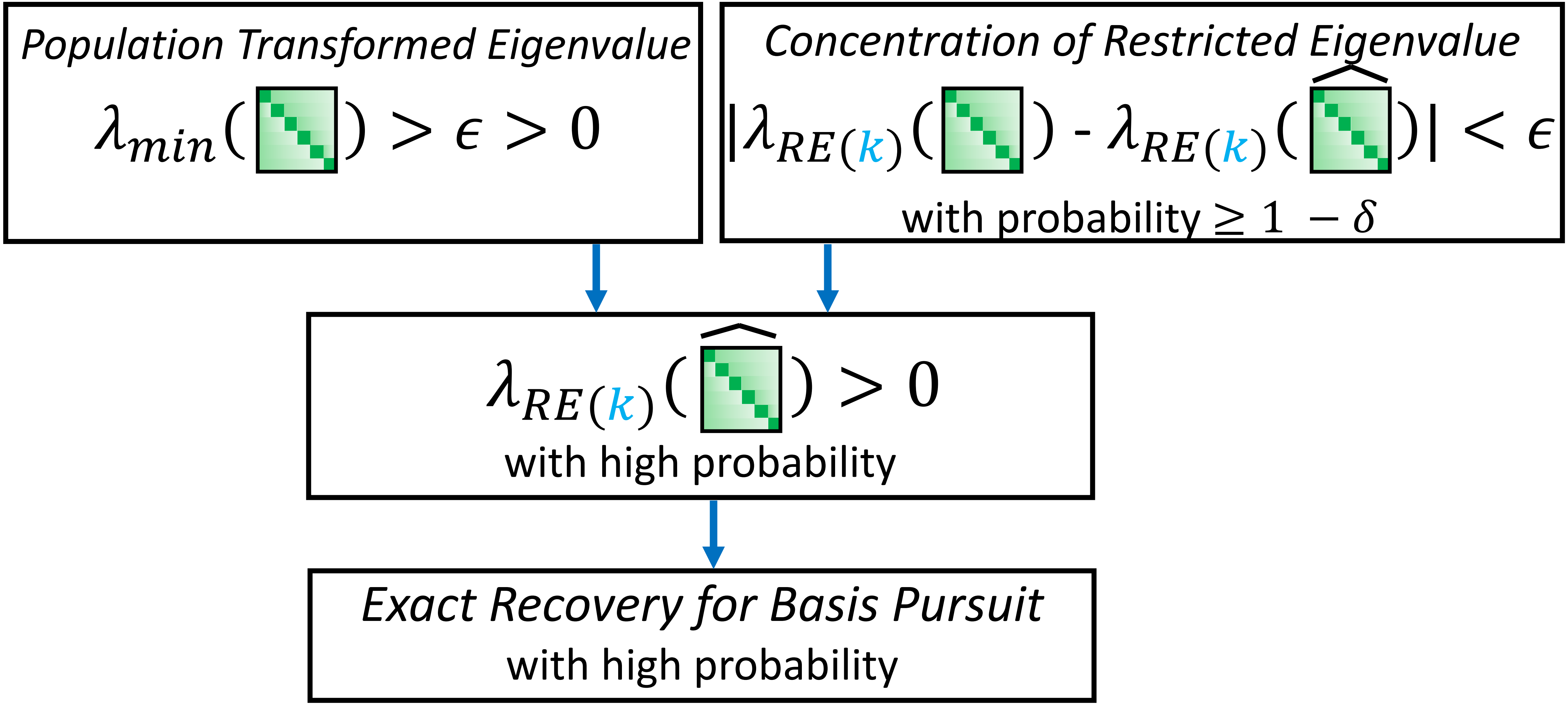


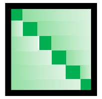
$$C = \{v \colon ||v_S||_1 \geq ||v_{S^c}||_1\}$$

$$|S| = k$$

$= \mathbb{E}[\log|x| \log|x|^T]$

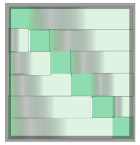
Sample Complexity Analysis



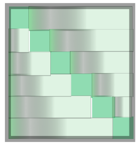
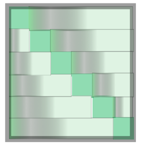
 $= \mathbb{E}[\log|x| \log|x|^T]$

Sample Complexity Analysis

Population Transformed Eigenvalue

$\lambda_{min}($  $) > \epsilon > 0$

Concentration of Restricted Eigenvalue

$|\lambda_{RE(k)}($  $) - \lambda_{RE(k)}(\widehat{}$  $)| < \epsilon$

ility $\geq 1 - \delta$

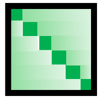
Sample Complexity Bound:

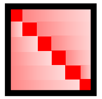
$$m = O\left(\frac{k^2 \log 2k}{1-\rho} \cdot \log^2 \frac{2p}{\delta}\right)$$

with high probability

Exact Recovery for Basis Pursuit

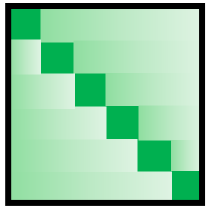
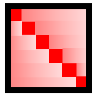
with high probability

 $= \mathbb{E}[\log|x| \log|x|^T]$

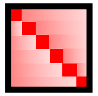
 $= \mathbb{E}[xx^T]$

Population Minimum Eigenvalue

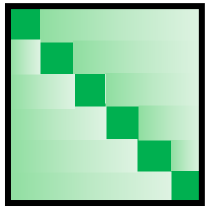
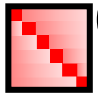
- Hermite expansion of $\log|\cdot|$:

 $= c_0^2 1_{pxp} + \sum_{l=1}^{\infty} c_{2l}^2$  $^{(2l)}$

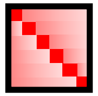
- $l \geq 1$: $c_{2l}^2 \sim \frac{\sqrt{\pi}}{4} \cdot \frac{1}{l^{3/2}}$

-  $^{(2l)}$ off-diagonals decay fast!

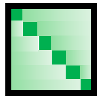
- Apply λ_{min} to Hermite formula:

λ_{min}  $\geq \sum_{l=1}^{\infty} c_{2l}^2 \lambda_{min}$  $^{(2l)}$

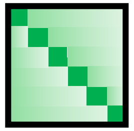
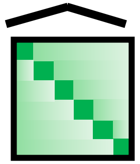
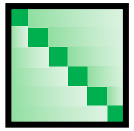
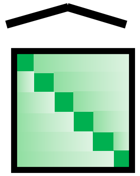
- Apply Gershgorin Circle Theorem:

λ_{min}  $^{(2l)} \geq 1 - (p-1)\rho^{2l}$

(for large enough l)

 $= \mathbb{E}[\log|x| \log|x|^T]$

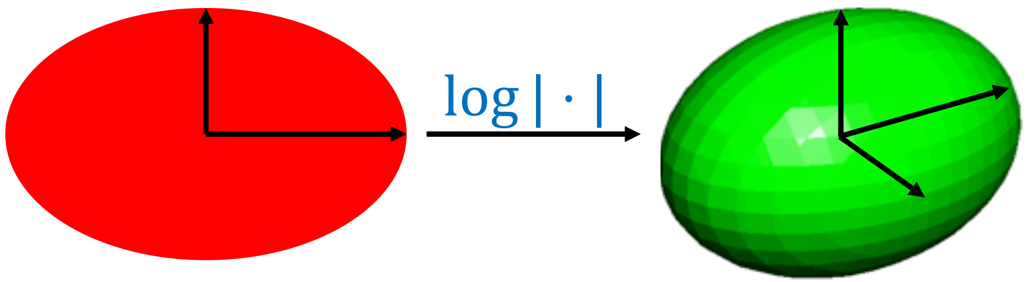
Concentration of Restricted Eigenvalue

- $|\lambda_{RE(k)}($  $) - \lambda_{RE(k)}($  $)| < k \cdot ||$  $-$  $||_\infty$
- Log-transformed variables are **sub-exponential**
- Elementwise ℓ_∞ error concentrates [Kuchibhotla & Chakrabortty '18]

4. Conclusion

Recap

- Attribute-efficient algorithm for **monomials**
  - Prior (nonlinear) work: **uncorrelated** features
  - This work: allow highly **correlated** features
    - Works beyond multilinear monomials
- Blessing of nonlinearity



Future Work

- Rotations of product distributions
- Additive noise
- Sparse polynomials with correlated features

Thanks! Questions?