Lecturers: Kiran Vodrahalli and Mikhail Khodak

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1 Introduction and Problem Setup

We give a presentation of the paper "On Lipschitz Extension From Finite Subsets", by Assaf Naor and Yuval Rabani, (2015). For the convenience of the reader referencing the original paper, we have kept the numbering of the lemmas the same.

Consider the setup where we have a metric space (X, d_X) and a Banach space $(Z, \|\cdot\|_Z)$. For a subset $S \subseteq X$, consider a 1-Lipschitz function $f : S \to Z$. Our goal is to extend f to $F : X \to Z$ without experiencing too much growth in the Lipschitz constant $\|F\|_{Lip}$ over $\|f\|_{Lip}$.

Definition 1.1. $e(X, S, Z)$ and its variants.
Define $e(X, S, Z)$ to be the infimum over the sequence of K satisfying $\|F\|_{Lip} \leq K\|f\|_{Lip}$ (i.e., $e(X, S, Z)$ is the least upper bound for $\frac{\|F\|_{Lip}}{\|f\|_{Lip}}$ for a particular S, X, Z).

Then, define $e(X, Z)$ to be the supremum over all subsets S for $e(X, S, Z)$: So of all subsets, what's the largest least upper bound for the ratio of Lipschitz constants?

We may also want to consider supremums over $e(X, S, Z)$ for S with a fixed size. We can formulate this in two ways. $e_n(X, Z)$ is the supremum of $e(X, S, Z)$ over all S such that $|S| = n$. We can also describe $e_\epsilon(X, Z)$ as the supremum of $e(X, S, Z)$ over all S which are ϵ-discrete in the sense that $d_X(x, y) \geq \epsilon \cdot \text{diam}(S)$ for distinct $x, y \in S$ and some $\epsilon \in [0, 1]$.

Definition 1.2. Absolute extendability.
We define $\mathbf{ae}(n)$ to be the supremum of $e_n(X, Z)$ over all possible metric spaces (X, d_X) and Banach spaces $(Z, \|\cdot\|_Z)$. Identically, $\mathbf{ae}(\epsilon)$ is the supremum of $e_\epsilon(X, Z)$ over all (X, d_X) and $(Z, \|\cdot\|_Z)$.

From now on, we will primarily discuss subsets $S \subseteq X$ with size $|S| = n$. Bounding the supremum, the absolute extendability $\mathbf{ae}(n) < K$ allows us to make general claims about the extendability of maps from metric spaces into Banach spaces. **Any** Banach-space valued 1-Lipschitz function defined on metric space (M, d_M) can therefore be extended to **any** metric space M' such that M' contains M (up to isometry; as long as you can embed M in M' with an injective distance preserving map) such that the Lipschitz constant of the extension is at most K.

Therefore, for the last thirty years, it has been of interest to understand upper and lower bounds on $\mathbf{ae}(n)$, as we want to understand the asymptotic behavior as $n \to \infty$. In the 1980s, the following upper and lower bound were given by Johnson and Lindenstrauss and Schechtman:

$$\sqrt{\frac{\log n}{\log \log n}} \lesssim \mathbf{ae}(n) \lesssim \log n$$

In 2005, the upper bound was improved:

$$\sqrt{\frac{\log n}{\log\log n}} \lesssim \mathbf{ae}(n) \lesssim \frac{\log n}{\log\log n}$$

In this talk, we improve the lower bound for the first time since 1984 to

$$\sqrt{\log n} \lesssim \mathbf{ae}(n) \lesssim \frac{\log n}{\log\log n}$$

1.1 Why do we care?

This improvement is of interest primarily not because of the removal of a $\sqrt{\log\log n}$ term in the denominator. It is due to the fact that the approach taken to get the lower bound provided by Johnson-Lindenstrauss 1984 has an inherent limitation. The approach of Johnson-Lindenstrauss to get the lower bound is to prove the nonexistance of linear projections of small norm. By considering a specific case for f, X, S, Z, we can get a lower bound on $\mathbf{ae}(n)$. Consider a Banach space $(W, \|\cdot\|_W)$ and let $Y \subseteq W$ be a k-dimensional linear subspace of W with N_ϵ an ϵ-net in the unit sphere of Y, and then define $S_\epsilon = N_\epsilon \cup \{0\}$. Fix $\epsilon \in (0, 1/2)$. We take $f : S_\epsilon \to Y$ to be the identity mapping, and wish to find an extension to $F : W \to Y$. Then, in our setup, we let $X = W$, $S = S_\epsilon$, $Z = Y$. We seek to bound the magnitude of the Lipschitz constant of F, call it L. Johnson-Lindenstrauss prove that for $\epsilon \lesssim \frac{1}{k^2}$, there **exists** a linear projection $P : W \to Y$ with $\|P\| \lesssim L$. We can now proceed to lower bound L by lower-bounding $\|P\|$ for all P. The classical Kadec'-Snobar theorem says that there always exists a projection with $\|P\| \leq \sqrt{k}$. Therefore, the best (largest) possible lower bound we could get will be $L \gtrsim \sqrt{k}$ by Kadec'-Snobar. But this is bad:

Taking $n = |S_\epsilon|$, by bounds on ϵ-nets we get $k \asymp \frac{\log n}{\log(1/\epsilon)}$ which implies

$$L \gtrsim \sqrt{\frac{\log n}{\log(1/\epsilon)}}$$

In order to get the lower bound on $\mathbf{ae}(n)$ of $\sqrt{\log n}$, we must take ϵ to be a universal constant. However, from a lemma by Benyamini (in our current setting), $L \lesssim e_\epsilon(X, Z) \lesssim 1/\epsilon = O(1)$, which means that any lower bound we get on L will be too small (and won't even tend to ∞). Therefore, we must make use of nonlinear theory to get the $\sqrt{n}$ lower bound on $\mathbf{ae}(n)$.

1.2 Outline of the Approach

Let us formally state the theorem, and then give the approach to the proof.

Theorem 1.3. *Theorem* 1.
For every $n \in \mathbb{N}$ *we have* $\boldsymbol{ae}(n) \gtrsim \sqrt{\log n}$.

We give a metric space X, a Banach space Z, a subset $S \subseteq X$, a function $f : S \to Z$ such that f extends to $F : X \to Z$ where $\|F\|_{Lip} \leq K\|f\|_{Lip}$.

Let V_G be the vertices of a finite graph G with distance metric the shortest path metric d_G where edges all have length 1.

We define our metric space $X = (V_G, d_{G_r(S)})$ where $G_r(S)$ is the r-magnification of the shortest path metric on V_G. S is an n-vertex subset $(S, d_{G_r(S)})$. Our Banach space $Z = (\mathbb{R}_0^X, \|\cdot\|_{W_1(X, d_{G_r(S)})})$ is equipped with the Wasserstein-1 norm induced by the r-magnification of the shortest path metric on the graph. Note that $\mathbb{R}_0^X$ is just weight distributions on the vertices of X which sum to zero in the image. Our $f : S \to \mathbb{R}_0^S \subseteq Z$, and we extend to $F : X \to Z$. We will show how to choose r and $|S|$ optimally to get the result.

The rest of my section of the talk will give the requisite definitions and lemmas to understand the full proof.

2 r-Magnification

Definition 2.1. r-magnification of a metric space.
Given metric space (X, d_X) and $r > 0$, for every subset $S \subseteq X$ we define $X_r(S)$ as a metric space on the

points of X equipped with the following metric:

$$d_{X_r(S)}(x,y) = d_X(x,y) + r|\{x,y\} \cap S|$$

and where $d_{X_r(S)}(x,x) = 0$. All this is saying is that when we have distinct points $x, y \in S$, we have the metric is $2r + d_X(x,y)$, when one point is in S and one point is outside, we have $r + d_X(x,y)$, and when both x, y are outside, the metric is unchanged.

The significance of this definition is as follows: It's easier for functions on S to be Lipschitz (we enlarge the denominator) without affecting functions on $X \setminus S$. Thus, there are more potential f we can draw from which satisfy 1-Lipschitzness which can have potentially large Lipschitz extensions (i.e., large K) since we don't make it easier to be Lipschitz on $X \setminus S$ (which we must deal with in the extension space).

However, we can't make r too large: the minimum distance between x, y in S becomes close to $\mathrm{diam}(S)$ under r-magnification as r increases. Let us assume the minimum distance between x, y is 1 (as it would be in an undirected graph with an edge between x, y under the shortest path metric). Particularly, for distinct $x, y \in S$, since $\mathrm{diam}(S, d_{X_r(S)}) = 2r + \mathrm{diam}(S, d_X)$,

$$d_{X_r(S)}(x,y) \geq 2r + 1 = \frac{2r+1}{2r + \mathrm{diam}(S, d_X)} \cdot \mathrm{diam}(S, d_{X_r(S)})$$

Then recall that $e_\epsilon(X, Z)$ is the supremum over S such that are ϵ-discrete, where here, $\epsilon = \frac{2r+1}{2r+\mathrm{diam}(S,d_X)}$. Earlier we saw a bound that

$$e_\epsilon(X,Z) \lesssim 1/\epsilon = \frac{2r + \mathrm{diam}(S, d_X)}{2r+1} \leq 1 + \frac{\mathrm{diam}(S, d_X)}{r}$$

Thus, if we make r too large, we again are bounding $e_\epsilon(X, Z) \lesssim 1 = O(1)$, which means our choice of X and Z is not good to get a large lower bound (again, we're not even going to ∞).

Thus we must balance our choice of r appropriately.

3 Wasserstein-1 norm

Now we come to the second part of our choice of Z. Note that we will define $\mathbb{R}_0^X$ to be the set of functions on the points of X such that for each $f \in \mathbb{R}_0^X$, $\sum_{x \in X} f(x) = 0$. We use e_x to denote the indicator weight map with 1 at point x and 0 everywhere else.

Definition 3.1. Wasserstein-1 Norm.
The Wasserstein-1 norm is the norm induced by the following origin-symmetric convex body in finite metric space (X, d_X):

$$K_{(X,d_X)} = \mathrm{conv}\left\{ \frac{e_x - e_y}{d_X(x,y)} : x, y \in X, x \neq y \right\}$$

This is a unit ball on $\mathbb{R}_0^X$. We denote the induced norm by $\|\cdot\|_{W_1(X,d_X)}$.

We can give an equivalent (proven with the Kantorovich-Rubinstein duality theorem) definition of the Wasserstein-1 distance:

Definition 3.2. Wasserstein-1 distance and norm.
Let $\Pi(\mu, \nu)$ be all measures on π on $X \times X$ such that

$$\sum_{y \in X} \pi(y,z) = \nu(z)$$

for all $z \in X$ and

$$\sum_{z \in X} \pi(y,z) = \mu(y)$$

for all $y \in X$. Then, the Wasserstein-1 distance (earthmover) is

$$W_1^{d_X}(\mu,\nu) = \inf_{\pi \in \Pi(\mu,\nu)} \sum_{x,y \in X} d_X(x,y)\pi(x,y)$$

In the case that $\mu = \nu$, we automatically have $(\mu \times \nu)/\mu(X) \in \Pi(\mu, \nu)$ (normalizing by one measure trivially gives the other), so Π is nonempty. The norm induced by this metric for $f \in \mathbb{R}_0^X$ is

$$\|f\|_{W_1(X,d_X)} = W_1^{d_X}(f^+, f^-)$$

where $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$ (since $\sum_{x \in X} f(x) = 0$, we need to make sure that μ, ν are both nonnegative measure). Futhermore, we have $\sum_{x \in X} f^+(x) = \sum_{x \in X} f^-(x)$ (same total mass), which means that $\Pi(f^+, f^-)$ is nonempty.

Definition 3.3. $\ell_1(X)$ norm.
Note that using standard notation, we can also define an ℓ_1 norm on f by

$$\|f\|_{\ell_1(X)} = \sum_{x \in X} |f(x)| = \sum_{x \in X} f^+(x) + f^-(x)$$

Thus, for our restrictions on f, $\sum_{x \in X} f^+(x) = \sum_{x \in X} f^-(x) = \|f\|_{\ell_1(X)}/2$.

Now we give a simple lemma which gives bounds for the Wasserstein-1 norm induced by the r-magnification of a metric on X.

Lemma 3.4. *Lemma 7.*
For (X, d_X) a finite metric space, we have

1. $\|e_x - e_y\|_{W_1(X,d_X)} = d_X(x, y)$ *for every $x, y \in X$.*
2. *For all $f \in \mathbb{R}_0^X$,*

$$\frac{1}{2} \min_{x,y \in X; x \neq y} d_X(x, y) \|f\|_{\ell_1(X)} \leq \|f\|_{W_1(X,d_X)} \leq \frac{1}{2} \mathrm{diam}(X, d_X) \|f\|_{\ell_1(X)}$$

3. *For every $r > 0$, $S \subseteq X$, for all $f \in \mathbb{R}_0^S$,*

$$r\|f\|_{\ell_1(S)} \leq \|f\|_{W_1(S,d_{X_r(S)})} \leq \left(r + \frac{\mathrm{diam}(X, d_X)}{2}\right) \|f\|_{\ell_1(S)}$$

Proof.

1. This follows directly from the unit ball interpretation of the Wasserstein-1 norm, since $\frac{e_x - e_y}{d_X(x,y)}$ is by the first definition on the unit ball.
2. Let $m = \min_{x,y} d_X(x, y) > 0$. For distinct $x, y \in X$, we have

$$\max_{x,y \in X; x \neq y} \left\| \frac{e_x - e_y}{d_X(x, y)} \right\|_{\ell_1(X)} \leq \max_{x,y \in X; x \neq y} \frac{\|e_x - e_y\|_{\ell_1(X)}}{m} = \frac{2}{m}$$

   since $0 < m \leq d_X(x, y)$ and $1 + 1 = 2$. Therefore $\frac{e_x - e_y}{d_X(x,y)} \in \frac{2}{m} B_{\ell_1(X)}$. These elements span K_{X,d_X}, so we have $K_{X,d_X} \subseteq \frac{2}{m} B_{\ell_1(X)}$ and we get the first inequality. The second inequality follows from

$$\|f\|_{W_1(X,d_X)} = \inf_{\pi \in \Pi(f^+, f^-)} \sum_{x,y \in X} d_X(x, y)\pi(x, y) \leq \mathrm{diam}(X, d_X) \sum_{x \in X} f^+(x) = \mathrm{diam}(X, d_X)\|f\|_{\ell_1(X)}/2$$

3. This inequality is a special case of the previous inequality. We have that for $X_r(S)$, $m \geq 2r$ (so $\frac{2}{m} \leq \frac{1}{r}$) and $\frac{1}{2}\mathrm{diam}(X, d_{X_r(S)}) \leq \frac{1}{2}(2r + \mathrm{diam}(X, d_X)) = \left(r + \frac{\mathrm{diam}(X,d_X)}{2}\right)$. Plugging in these estimates give the inequality.

□

4 Graph theoretic lemmas

4.1 Expanders

We will need several properties of edge-expanders in our proof of the main theorem. For this section, we fix $n, d \geq 3$ and let G be a connected n-vertex d-regular graph. We can imagine $d = 3$ in this section, all that matters is that d is fixed.

First we record two basic average bounds on distance in the shortest-path metric, denoted d_G.

Lemma 4.1. *Average shortest-path metric and r-magnified average shortest-path bounds.*

1. *d_G lower bound: For nonempty $S \subseteq V_G$,*

$$\frac{1}{|S|^2} \sum_{x,y \in S} d_G(x,y) \geq \frac{\log |S|}{4 \log d}$$

2. *$d_{G_r(S)}$ equality: For some $S \subseteq V_G$ and $r > 0$,*

$$\frac{1}{|E_G|} \sum_{(x,y) \in E_G} d_{G_r(x,y)} = 1 + \frac{2r|S|}{n}$$

Proof. 1. The smallest nonzero distance in G is at least 1. Thus, the average is bounded below by $\frac{1}{|S|^2}|S|(|S|-1) = 1 - \frac{1}{|S|}$ since G is connected (shortest case is complete graph on n vertices). Then, $1 - 1/a \geq (\log a)/4 \log 3$ for $a \in [15]$ ($d = 3$ maximizes), so we proceed assuming $|S| \geq 16$. Let's bound the distance in the average. Since G is d-regular, for every $x \in V_G$ the number of vertices y such that $d_G(x,y) \leq k-1$ is at most $\sum_{i=0}^{k-1} d^i$. The rest of the vertices are farther away. Since $1+\cdots+d^{k-2} < d^{k-1}$ we have $\#\{y : d_G(x,y) \leq k-1\} \leq 2d^{k-1}$. Choosing $k = 1 + \lfloor \log_d(|S|/4) \rfloor$ gives that $2d^{k-1} \leq \frac{|S|}{2}$. Therefore

$$\frac{1}{|S|^2} \sum_{x,y \in S} d_G(x,y) \geq \frac{1}{|S|^2} * |S| * |S|/2 * (k-1) = \frac{k-1}{2} = \frac{\log(|S|/4)}{2 \log d} \geq \frac{\log |S|}{4 \log d}$$

since $|S| \geq 16$.

2. Let E_1 be edges completely contained in S and E_2 be edges partially contained in S. Because G is d-regular, $2|E_1| + |E_2| = d|S|$ (2 vertices in S for E_1, only 1 vertex for E_2, then divide by d for overcounting since each vertex in S hits d other vertices, and we count exactly the edges which have at least one vertex in S). Note that $|E_G| = dn/2$ by double-counting vertices. Then for each edge in E_1 we add $2r$, for each edge in E_2 we add r, and otherwise we add 0 to the base distance of an edge, which is 1. Therefore,

$$\begin{aligned} \frac{1}{|E_G|} \sum_{(x,y) \in E_G} d_{G_r(x,y)} &= \frac{((0+1)|E_G \setminus (E_1 \cup E_2)| + (r+1)|E_2| + (2r+1)|E_1|)}{|E_G|} \\ &= 1 + \frac{r(2|E_1| + |E_2|)}{|E_G|} = 1 + \frac{rd|S|}{dn/2} = 1 + \frac{2r|S|}{n} \end{aligned}$$

□

Now we introduce the definition of edge expansion.

Definition 4.2. Edge expansion $\phi(G)$.
G is a connected n-vertex d-regular graph. Consider $S, T \subseteq V_G$ disjoint subsets. Let $E_G(S,T) \subseteq E_G$ denote the set of edges which bridge S and T. Then the **edge-expansion** $\phi(G)$ is defined by

$$\phi(G) = \sup \left\{ \phi : |E_G(S, V_G \setminus S)| \geq \phi \frac{|S|(n-|S|)}{n^2} |E_G|, \forall S \subseteq V_G, \phi \in [0, \infty) \right\}$$

We give an equivalent formulation of edge expansion via the cut-cone decomposition:

Lemma 4.3. *Edge-Expansion: Cut-cone Decomposition of Subsets of ℓ_1.*
$\phi(G)$ is the largest ϕ such that for all $h : V_G \to \ell_1$,

$$\frac{\phi}{n^2} \sum_{x,y \in V_G} \|h(x) - h(y)\|_1 \leq \frac{1}{|E_G|} \sum_{(x,y) \in E_G} \|h(x) - h(y)\|_1$$

Proof. We will assume this for this talk. □

Now we combine Lemma 7 and the cut-cone decomposition to get Lemma 8:

Lemma 4.4. *Lemma* 8.
Fix $n \in \mathbb{N}$ and $\phi \in (0, 1]$. Suppose G is an n-vertex graph with edge expansion $\phi(G) \geq \phi$. For all nonempty $S \subseteq V_G$ and $r > 0$, every $F : V_G \to \mathbb{R}_0^S$ satisfies

$$\frac{1}{n^2} \sum_{x,y \in V_G} \|F(x) - F(y)\|_{W_1(S, d_{G_r(S)})} \leq \frac{2r + \operatorname{diam}(S, d_G)}{(2r+1)\phi} \cdot \frac{1}{|E_G|} \sum_{(x,y) \in E_G} \|F(x) - F(y)\|_{W_1(S, d_{G_r(S)})}$$

Basically, whereas before we were bounding average norms (normal and r-magnified) in the domain space V_G, we are now bounding average norms in the image space using the Wasserstein-1 norm induced by the r-magnification of the shortest path metric on the graph.

Proof. First, plug in $h = F$ into the cut-cone decomposition, where the norm is now defined by $\ell_1(S)$. Then, we know that $\operatorname{diam}(S, d_{G_r}(S)) = 2r + \operatorname{diam}(S, d_G)$ and the smallest positive distance in $(S, d_{G_r(S)})$ is $2r + 1$. Applying Lemma 7, every $x, y \in V_G$ satisfy

$$\frac{2r+1}{2} \|F(x) - F(y)\|_{\ell_1(S)} \leq \|F(x) - F(y)\|_{W_1(S, d_{G_r(S)})} \leq \frac{2r + \operatorname{diam}(S, d_G)}{2} \|F(x) - F(y)\|_{\ell_1(S)}$$

Plugging these estimates directly into the cut-cone decomposition

$$\frac{1}{n^2} \sum_{x,y \in V_G} \|F(x) - F(y)\|_{\ell_1(S)} \leq \frac{1}{\phi |E_G|} \sum_{(x,y) \in E_G} \|F(x) - F(y)\|_{\ell_1(S)}$$

gives the desired result. □

4.2 An Application of Menger's Theorem

Here, we want to bound below the number of edge-disjoint paths.

Lemma 4.5. *Lemma* 9.
Let G be an n-vertex graph and $A, B \subseteq V_G$ be disjoint. Fix $\phi \in (0, \infty)$ and suppose $\phi(G) \geq \phi$. Then

$$\#\{\text{edge-disjoint paths joining } A \text{ and } B\} \geq \frac{\phi |E_G|}{2n} \cdot \min\{|A|, |B|\}$$

Proof. Let m be the maximal number of edge-disjoint paths joining A and B. By Menger's theorem from classical graph theory, there exists a subset of edges $E^* \subseteq E_G$ with $|E^*| = m$ such that every path in G joining $a \in A$ to $b \in B$ contains an edge from E^*.

Now consider the graph $G^* = (V_G, E_G \setminus E^*)$. In this graph, there are no paths between A and B. Now, if we let $C \subseteq V_G$ be the union of all connected components of G^* containing an element of A, then $A \subseteq C$ and $B \cap C = \emptyset$. Since we covered the maximal possible vertices reachable from A with C, all edges between C and $V_G \setminus C$ are in E^*. Therefore, $|E_G(C, V_G \setminus C)| \leq |E^*| = m$. By the definition of expansion,

$$m \geq |E_G(C, V_G \setminus C)| \geq \phi \frac{\max\{|C|, n - |C|\} \cdot \min\{|C|, n - |C|\}}{n^2} |E_G| \geq \frac{\phi \min\{|A|, |B|\} \cdot |E_G|}{2n}$$

since $\max\{|C|, n - |C|\} \geq n/2$ and since $A \subseteq C, B \subseteq V_G \setminus C$, we have $\min\{|C|, n - |C|\} \geq \min\{|A|, |B|\}$. □

5 Main Proof

We fix $d, n \in \mathbb{N}, \phi \in (0,1)$ and let G be a d-regular graph on n vertices with $\phi(G) \geq \phi$. We also fix a nonempty subset $S \subset V_G$ and $r > 0$ and define a mapping

$$f : (S, d_{G_r(S)}) \mapsto \left(\mathbb{R}_0^S, \|\cdot\|_{W_1(S, d_{G_r(S)})}\right) \quad \text{s.t.} \quad f(x) = e_x - \frac{1}{|S|}\sum_{z \in S} e_z \ \ \forall \ x \in S$$

Then suppose we have that some $F : V_G \mapsto \mathbb{R}_0^S$ extends f and for some $L \in (0,\infty)$ we have

$$\|F(x) - F(y)\|_{W_1(S, d_{G_r(S)})} \leq L d_{G_r(S)}(x,y)$$

For $x \in V_G, s \in (0,\infty)$ define

$$B_S(x) = \{y \in V_G : \|F(x) - F(y)\|_{W_1(S, d_{G_r(S)})} \leq s\}$$

i.e.e B_s is the inverse image of F of the ball (in the Wasserstein 1-norm) of radius s centered at $F(x)$. By the lemma consequence of Menger's Theorem, since $\phi \geq \phi(G)$ we have

$$m \geq \frac{\phi d}{4}\min\{|S \backslash B_s(x)|, |B_s(x)|\} \qquad 1$$

for m edge-disjoint paths between $S\backslash B_s(x)$ and $B_s(x)$, i.e. we can find indices $k_1, \ldots, k_m \in \mathbb{N}$ and vertex sets $\{z_{j,1}, \ldots, z_{j,k_j}\}_{j=1}^m \in V_G$ s.t. $\{z_{j,1}\}_{j=1}^m \subset S\backslash B_s(x)$, $\{z_{j,k_j}\}_{j=1}^m \subset B_s(x)$ (i.e. the beginnings and ends of paths are in different disjoint subsets) and such that $\{\{z_{j,1}, z_{j,i+1} : j \in \{1,\ldots,m\} \wedge i \in \{1, \ldots, k_j - 1\}\}$ are distinct edges in E_G (i.e. edge-disjointedness). Now take an index subset $J \subset \{1, \ldots, m\}$ s.t. $\{z_{j,1}\}_{j \in J}$ are distinct and $\{z_{j,1}\}_{j \in J} = \{z_{i,1}\}_{i=1}^m$. For $j \in J$ denote by d_j the number of $i \in \{1, \ldots, m\}$ for which $z_{j,1} = z_{i,1}$. Then since G is d-regular and $\{\{z_{i,1}, z_{i,2}\}\}_{i=1}^m$ are distinct, $\max\limits_{j \in J} d_j \leq d$. Since $\sum\limits_{j \in J} d_j = m$, from 1 we have that

$$|J| \geq \frac{m}{d} \geq \frac{\phi}{4}\min\{|S\backslash B_s(x)|, |B_s(x)|\} \qquad 2$$

The quantity $|J|$ can be upperbounded as follows:

Lemma 5.1. *Lemma* 10*:* $|J| \leq \max\left\{d^{16(s-r)}, \frac{16nLd\log d}{\log n}\left(1 + \frac{2r|S|}{n}\right)\right\}$

Proof. Assume $|J| \leq d^{16(s-r)}$ (otherwise we are done). This is equivalent to

$$s - r < \frac{\log|J|}{16 \log d}$$

Now since $\{z_{j,1}\}_{j \in J} \subset S$ and $F(x) = f(x) \ \forall \ x \in S$ and is an isometry on $(S, d_{G_r(S)})$, by the definition of the r-magnified metric

$$\|F(z_{i,1}) - F(z_{j,1})\|_{W_1(S, d_{G_r(S)})} = d_{G_r(S)}(z_{i,1}, z_{j,1}) = 2r + d_G(z_{i,1}, z_{j,1})$$

This gives us

$$\begin{aligned}\sum_{j \in J}\|F(z_{j,1}) - F(z_{j,k_j})\|_{W_1(S, d_{G_r(S)})} &= \frac{1}{2|J|}\sum_{i,j \in J}\left(\|F(z_{i,1}) - F(z_{i,k_i})\|_{W_1(S, d_{G_r(S)})} + \|F(z_{j,1}) - F(z_{j,k_j})\|_{W_1(S, d_{G_r(S)})}\right) \\ &\geq \frac{1}{2|J|}\sum_{i,j \in J}\left(\|F(z_{i,1}) - F(z_{j,1})\|_{W_1(S, d_{G_r(S)})} + \|F(z_{z_i,k_i}) - F(z_{j,k_j})\|_{W_1(S, d_{G_r(S)})}\right)\end{aligned}$$

Since $\{z_{j,k_j}\}_{j \in J} \subset B_s(x)$, by the definition of $B_s(x)$ we have

$$\|F(z_{i,k_i}) - F(z_{j,k_j})\|_{W_1(S, d_{G_r(S)})} \leq \|F(z_{i,k_i}) - F(x)\|_{W_1(S, d_{G_r(S)})} + \|F(x) - F(z_{j,k_j})\|_{W_1(S, d_{G_r(S)})} \leq 2s \ \ \forall \ i,j \in J$$

so the previous inequality can be further continued as

$$\sum_{j\in J}\|F(z_{j,1})-F(z_{j,k_j})\|_{W_1(S,d_{G_r(S)})} \geq \frac{1}{2|J|}\sum_{i,j\in J} d_G(z_{i,1},z_{j,1})-(s-r)|J| \geq \frac{|J|\log|J|}{8\log d}-(s-r)|J| > \frac{|J|\log|J|}{16\log d}$$

where the second inequality above is a property of the expander graph. The same quantity can be bounded from above using the Lipschitz condition

$$\sum_{j\in J}\|F(z_{j,1})-F(z_{j,k_j})\|_{W_1(S,d_{G_r(S)})} \leq L\sum_{j\in J} d_{G_r(S)}(z_{j,1},z_{j,k_j}) \leq L\sum_{j\in J}\sum_{i=1}^{k_j-1} d_{G_r(S)}(z_{j,1},z_{j,i+1})$$

By the edge-disjointedness of the paths (specifically since $\{\{z_{j,1},z_{j,i+1}\} : j\in J \wedge i\in\{1,\dots,k_j-1\}\}$) are distinct edges in E_G, we have that

$$\sum_{j\in J}\sum_{i=1}^{k_j-1} d_{G_r(S)}(z_{j,1},z_{j,i+1}) \leq \sum_{\{u,v\}\in E_G} d_{G_r(S)}(u,v) = \frac{nd}{2}\left(1+\frac{2r|S|}{n}\right)$$

Everything together gives

$$\frac{Lnd}{2}\left(1+\frac{2r|S|}{n}\right) \geq \frac{|J|\log|J|}{16\log d}$$

By the simple fact that $a\log a\leq b \implies a\leq\frac{2b}{\log b}$ for $a\in[1,\infty), b\in(1,\infty)$, using $a=|J|$ and $b=8Lnd\log d\left(1+\frac{2r|S|}{n}\right)\geq n$ we have that

$$|J| \leq \frac{16nLd\log d}{\log n}\left(1+\frac{2r|S|}{n}\right)$$

completing the proof. ☐

The lemma has two corollaries, both depending on the following condition:

$$d^{16(s-r)} \leq \frac{\phi|S|}{8} \qquad L\leq\frac{\phi|S|\log n}{128\left(1+\frac{2r|S|}{n}\right)nd\log d} \qquad (3)$$

Corollary 5.2. *Corollary 11:* $\max_{x\in V_G}|B_s(x)| < \frac{|S|}{2}$

Proof. Pick an $x\in V_G$. If $B_s(x)\cap S$ is nonempty, then again using the standard estimate on expander graphs we have that $\exists\ y,z\in B_s(x)\cap S$ s.t.

$$d_G(y,z)\geq\frac{\log|B_s(x)\cap S|}{4\log d}$$

Since $y,z\in S$ and $F(x)=f(x)\ \forall\ x\in S$ and furthermore F is an isometry on $(S,d_{G_r(S)})$, we have similarly to before that

$$d_G(y,z)+2r = d_{G_r(S)}(y,z) = \|F(y)-F(z)\|_{W_1(S,d_{G_r(S)})} \leq \|F(y)-F(x)\|_{W_1(S,d_{G_r(S)})}+\|F(x)-F(z)\|_{W_1(S,d_{G_r(S)})} \leq 2s$$

where we have used first the triangle inequality and second the fact that $y,z\in B_s(x)$. Then using the first inequality in the proof we have

$$|B_s(x)\cap S| \leq d^{8(s-r)} \leq \sqrt{\frac{\phi|S|}{8}} \leq \frac{2|S|}{5}$$

where the second inequality comes from the first assumption. Now the above inequality implies that $|S\backslash B_s(x)|\geq\frac{3|S|}{5}$, which combined with Lemma 10 yields

$$\min\left\{\frac{3|S|}{5},|B_s(x)|\right\} < \max\left\{\frac{4d^{16(s-r)}}{\phi},\frac{64nLd\log d}{\phi\log n}\left(1+\frac{2r|S|}{n}\right)\right\}$$

However, the two original assumptions together imply that $\frac{3|S|}{5}$ is in fact greater than either value in the maximum, so

$$|B_s(x)| \le \max\left\{\frac{4d^{16(s-r)}}{\phi}, \frac{64nLd\log d}{\phi\log n}\left(1+\frac{2r|S|}{n}\right)\right\} \le \frac{|S|}{2}$$

□

Corollary 5.3. *Corollary* 12*:* $L \ge \frac{\phi s}{2\left(1+\frac{diam(G,d_G)}{2r}\right)\left(1+\frac{2r|S|}{n}\right)}$

Proof. By the definition of $B_s(x)$ for $x \in V_G$ and $y \in V_G \backslash B_s(x)$ we have $\|F(x)-F(y)\|_{W_1(S,d_{G_r(S)})} > s$, so

$$\begin{aligned}\frac{1}{n^2}\sum_{x,y\in V_G}\|F(x)-F(y)\|_{W_1(S,d_{G_r(S)})} &\ge \frac{1}{n^2}\sum_{x\in V_G}\sum_{y\in V_G\backslash B_s(x)}\|F(x)-F(y)\|_{W_1(S,d_{G_r(S)})}\\ &\ge \frac{s}{n^2}\sum_{x\in V_G}(n-|B_s(x)|) \ge s\left(1-\frac{\max_{x\in V_G}|B_s(x)|}{n}\right) \ge \frac{s}{2}\end{aligned}$$

where the last inequality comes from Corollary 11. Then since Lemma 8 gives

$$\begin{aligned}\frac{1}{n^2}\sum_{x,y\in V_G}\|F(x)-F(y)\|_{W_1(S,d_{G_r(S)})} &\le \frac{2r+\mathrm{diam}(S,d_G)}{(2r+1)\phi|E_G|}\sum_{x,y\in V_G}\|F(x)-F(y)\|_{W_1(S,d_{G_r(S)})}\\ &\le \frac{L\left(1+\frac{\mathrm{diam}(G,d_G)}{2r}\right)}{\phi|E_G|}\sum_{\{x,y\}\in E_G} d_{G_r(S)}(x,y)\\ &= \frac{L\left(1+\frac{\mathrm{diam}(G,d_G)}{2r}\right)\left(1+\frac{2r|S|}{n}\right)}{\phi}\end{aligned}$$

where the ineqaulity on the second line is true since F is an isometry on (S, d_G) and from the Lipschitz constant of f and the equality is average length of the r-magnification of the expander graph G. Combining these inequalities with the previous ones yields the corollary. □

Theorem 5.4. *Theorem* 13*: if* $0 < r \le diam(G, d_G)$ *then*

$$L \ge_C \frac{\phi}{1+\frac{r|S|}{n}}\min\left\{\frac{|S|\log n}{nd\log d}, \frac{16r^2\log d + r\log\left(\frac{\phi|S|}{8}\right)}{diam(G,d_G)\log d}\right\}$$

Proof. Assume $16r\log d + \log\left(\frac{\phi|S|}{8}\right) > 0$ (otherwise we are done) and choose $s = r + \frac{\log\left(\frac{\phi|S|}{8}\right)}{16\log d}$ s.t. $s > 0$ and $d^{16(s-r)} = \frac{\phi|S|}{8}$. Then the first inequality in 3 is satisfied, so either the second fails and L thus has that expression as a lower bound, or both are satisfied and we have the lower bound in Corollary 12. □

Theorem 5.5. *Theorem* 1*: for every* $n \in \mathbb{N}$ *we have* $\boldsymbol{ae}(n) \ge_C \sqrt{\log n}$*.*

Proof. Substituting $\phi \asymp 1$ and $\mathrm{diam}(G, d_G) \asymp \frac{\log n}{\log d}$ (the $\asymp$ indicates asymptotically for large n), and using the assumption $0 < r \le \mathrm{diam}(G, d_G)$, the lower bound given in Theorem 13 becomes

$$L \ge_C \frac{1}{1+\frac{r|S|}{n}}\min\left\{\frac{|S|\log n}{nd\log d}, \frac{r(r\log d + \log|S|)}{\log n}\right\}$$

Taking $S \subset V_G$ s.t. $|S| = \left\lfloor\frac{n\sqrt{d\log d}}{\sqrt{\log n}}\right\rfloor$ (we must have n
ged^d to ensure $|S| \le n$) and $r \asymp \frac{\log d}{\sqrt{d\log d}}$, which gives a lower bound of $L \ge_C \frac{\sqrt{\log n}}{\sqrt{d\log d}}$. Therefore

$$e_{\left\lfloor\frac{n\sqrt{d\log d}}{\sqrt{\log n}}\right\rfloor}(G_r(S), W_1(S, d_{G_r(S)})) \ge_C \frac{\sqrt{\log n}}{\sqrt{d\log d}}$$

which completes the proof for fixed d. □